The use of generating functions for IMO-type problems - Atli FF:

Generally when we enumerate things, we have some form of structure on a set of atomic objects. For example a graph on vertices, permutation on objects, functions on a domain and so on. If we call the number of such atomic objects in the structure its size, another important feature is that the number of structures of a given size is finite. This gives rise to a sequence $a_0, a_1, a_2, \dots$ of integers that can be worked with in lieu of the original structure. We will denote such a sequence (a_i). These notions of structures on atoms and the series they give rise to can be made formal, but that is outside the scope of this material. For those interested *Analytic Combinatorics* defines these notions in terms of combinatorial classes and *Combinatorial Species and Tree-like Structures* defines them in terms of species, we will borrow ideas from both. But first we will define generating functions, flagrantly borrowing material from generatingfunctionology.

Definition. For a sequence (a_i) we will define its *ordinary generating function* as $A(x) = \sum_{k=0}^{\infty} a_k x^k$.

Definition. For a sequence (a_i) we will define its *exponential generating function* as $\hat{A}(x) = \sum_{k=0}^{\infty} \frac{a_k}{k!} x^k$.

These definitions may seem odd for those who have not seen them before, but my hope is that by the end their usefulness will be clear. We will generally shorten these as ogf and egf. Many operations on such series translate very naturally into the realm of combinatorics for example. But first we shall see how they can be used to solve recurrences.

Consider a sequence (a_i) such that $a_0 = 0$ and $a_{n+1} = 2a_n + 1$. If one calculates out the first few terms it's not hard to spot the pattern, but let's try to use the series $A(x)$ to solve it. The trick is to sum the given recurrence over all valid n. It then becomes $\sum_{k=0}^{\infty} a_{k+1}x^k = 2\sum_{k=0} a_k x^k + \sum_{k=0}^{\infty} x^k$. The first term on the right hand side is simply $2A(x)$. The second is a geometric series, so it is equal to $1/(1-x)$. But what about the left hand side? Well if we multiply it by x it becomes $\sum_{k=0}^{\infty} a_{k+1}x^{k+1}$. This is simply $A(x)$ with the index shifted, so since $a_0 = 0$ it is equal to $A(x)$. All in all we get $A(x)x^{-1} = 2A(x) + (1-x)^{-1}$. This can be solved to get $A(x) = x(2/(1-2x) - 1/(1-x))$. Both of those inner functions are geometric sums, so we can expand this to get $A(x) = \sum_{k=0}^{\infty}(2^k - 1)x^k$.

This was a lot of heavy machinery for a rather simple recurrence, but its strength is that it works on much more complex recurrences. Consider next $a_0 = 1$ and $a_{n+1} = 2a_n + n$. We can sum over this again, getting $\sum_{k=0}^{\infty} a_{k+1}x^k = 2\sum_{k=0}^{\infty} a_k x^k + \sum_{k=0}^{\infty} kx^k$. We already know how to deal with most of this, but what about the last term? Taking the derivative of the geometric series we get $\sum_{k=0}^{\infty} kx^{k-1} = 1/(1-x)^2$. So if we multiply through by x we get what we want. This time we have to subtract off a_0 on the left hand side before dividing by x is valid, so we get $(A(x)-1)/x = 2A(x) + x/(1-x)^2$. This can be solved to get $A(x) = 2/(1-2x) - 1/(1-x)^2$. Using the same series as above this gives us $A(x) = \sum_{k=0}^{\infty}(2^{k+1} - k - 1)x^k$.

In these two examples we have only used generating functions as algebraic tools, but their real strength lies in their combinatorial interpretation. That's why we have both ogfs and egfs. Informally the ogf encodes unlabeled structures while the egf encodes labeled structures. In this context we are distinguishing whether the atoms of the structure are different to one another. For example a permutation is a structure on labelled elements while a triangulation of a polygon would be a structure on unlabelled elements. We distinguish between 1 and 2 in the permutation, but don't distinguish two triangles in a triangulation.

The reason for this distinction lies in how the multiplication of the generating functions work out. Thus let us now lay out a long sequence of propositions detailing how the sequences and generating functions interact before moving onto examples.

Proposition (1)**.** If (a_i) has an ogf $A(x)$ and egf $\hat{A}(x)$ then (a_{i+h}) has the ogf $\frac{A(x) - a_0 - a_1 x - \dots - a_{h-1}x^{h-1}}{x^h}$ and egf $\partial_x^h \hat{A}(x)$.

Proposition (2)**.** If (a_i) has an ogf $A(x)$, egf $\hat{A}(x)$ and P is a polynomial then $(P(i)a_i)$ has the ogf $P(x\partial_x)A(x)$ and egf $P(x\partial_x)\hat{A}(x)$.

Proposition (3)**.** If $(a_i), (b_i)$ have ogfs $A(x), B(x)$ and egfs $\hat{A}(x), \hat{B}(x)$ then $(a_i \pm b_i)$ has the ogf $(A \pm B)(x)$ and egf $(\hat{A} \pm \hat{B})(x)$.

Proposition (4)**.** If $(a_i), (b_i)$ have ogfs $A(x), B(x)$ and egfs $\hat{A}(x), \hat{B}(x)$ then $(A \cdot B)(x)$ is the ogf of $(\sum_{j=0}^{i} a_j b_{i-j})$ and $(\hat{A} \cdot \hat{B})(x)$ is the egf of $(\sum_{j=0}^{i} \binom{i}{j} a_j b_{i-j})$.

Proposition (5)**.** If (a_i) has an ogf $A(x)$ and $\hat{A}(x)$ then $A(x)^k$ is the ogf of $(\sum_{n_1+n_2+\dots+n_k=i} a_{n_1}a_{n_2}\dots a_{n_k})$ and $\hat{A}(x)^k$ is the egf of $(\sum_{n_1+n_2+\dots+n_k=i} \frac{i!}{n_1!n_2!\dots n_k!} a_{n_1}a_{n_2}\dots a_{n_k})$

Proposition (6)**.** If $A(x) = \sum_{k=0}^{\infty} a_k x^k$ then $A(x)/(1-x) = \sum_{k=0}^{\infty}(\sum_{j=0}^{k} a_j)x^k$.

Proposition (7)**.** $A(x) = \sum_{k=0}^{\infty} a_k x^k$ has a multiplicative inverse iff $a_0 \neq 0$. If it has one its coefficients (b_i) can be computed via the formula $b_n = -a_0^{-1} \sum_{k \geq 1} a_k b_{n-k}$.

Proposition (8)**.** For $A(x) = \sum_{k=0}^{\infty} a_k x^k$ and $B(x) = \sum_{k=0}^{\infty} b_k x^k$ the composition $A(B(x))$ is well defined iff $A(x)$ is a polynomial or $b_0 = 0$.

Proposition (9)**.** $A(x) = \sum_{k=0}^{\infty} a_k x^k$ has a compositional $B(x) = \sum_{k=0}^{\infty} b_k x^k$ inverse iff $a_0 = 0$ and $a_1 \neq 0$. If it exists it satisfies $A(B(x)) = B(A(x)) = x$.

We consider two examples to highlight the difference in multiplication. Let (c_n) be the sequence counting the number of Dyck paths, that is to say paths from $(0,0)$ to $(2n,0)$ with steps $(1,-1)$ and $(1,1)$ and never dip below the x-axis. We consider the first return decomposition of a Dyck path. A Dyck path is either empty, or returns to the x-axis at some point after $(0,0)$. The path up until this return starts with an upward step and ends with a downward step, and does not dip below $x = 1$ in between. The part after this return is also another valid Dyck path. Thus we have decomposed our path into a single fixed step and two smaller Dyck paths of variable size. In terms of ogfs we can write this as $C(x) = xC(x)^2 + 1$. Solving for $C(x)$ this gives us $(1 + \pm\sqrt{1-4x})/2x$. The limit $x \to 0$ has to be 1 since $c_0 = 1$, so the $\pm$ is a minus. Using the extended binomial theorem and some calculations one can even use this to get $c_n = \binom{2n}{n}(n+1)^{-1}$. Here ogfs were applicable rather than egfs since we do not distinguish between the steps of a path.

Let $\hat{S}(x)$ be the egf of permutations. We get $\hat{S}(x) = \sum_{k=0}^{\infty} \frac{k!}{k!} x^k = (1-x)^{-1}$. Let $\hat{E}$ be the egf of sets, so there's one structure of any given size, so $\hat{E}(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$ (It's customary to denote it E because set is ensemble in French). Lastly let $\hat{D}$ be the egf of derangements. Then when creating a permutation we can choose a subset to be our fixed points, these have the structure of a set, and the rest is a derangement. In terms of generating functions this means $\hat{S}(x) = \hat{E}(x)\hat{D}(x)$. Since $e_0 = 1 \neq 0$ it's valid to say $\hat{D}(x) = \hat{S}(x)/\hat{E}(x) = e^{-x}(1-x)^{-1}$ by (7). Furthermore by (6) we get that $d_n/n! = 1/0! - 1/1! + 1/2! - \cdots + (-1)^n/n!$.

So combinatorially multiplication means creating some structure from one factor on a subset, and the structure from the other factor on the rest. What about composition? If A corresponds to some structure on a ground set of atoms and B corresponds to some other structure, $A \circ B$ will correspond to a structure where each of the atoms in the A-structure have been replaced by B-structures.

For example if we consider $x + x$, this corresponds to having two objects of size 1 and nothing else. If we create a sequence of these objects, we get something equivalent to binary words. That gives us that the ogf of binary words is $(1-x)^{-1} \circ (x+x) = (1-2x)^{-1} = \sum_{k=0}^{\infty} 2^k x^k$, which shouldn't be surprising. What about the generating function for integer compositions? A composition is a sequence of positive integers, so it should be given by $(1-x)^{-1}$ composed with $(1-x)^{-1} - 1$, where we subtract one in order to disallow structures of size zero. This comes out to $(1-x)/(1-2x) = 1 + \sum_{k=1}^{\infty} 2^{k-1} x^k$, which should also be an unsurprising result. Yet another example is the ogf of rooted unlabeled trees $T_R(x)$. How can we decompose such a structure? Well, a rooted unlabeled tree is just a root along with a set of subtrees, that is to say $T_R(x) = xE(T_R(x))$. This gives us $T_R(x) = (1 \pm \sqrt{1-4x})/2x$ and by the same argument as last time we can see that the $\pm$ is a minus, giving us that rooted unlabeled trees are enumerated by the catalan numbers.

What about labeled composition? Well, for example set partitions are simply sets of non-empty sets, so their egf is given by $(\hat{E} \circ (\hat{E} - 1))(x) = e^{e^x - 1}$. Here we must use egfs since the individual atoms, the elements of the sets, are distinguishable. For the next example we consider the structure of cycles. On labeled atoms there are $(n-1)!$ ways to arrange n of them into a cycle, so the exponential generating function is $\widehat{\text{Cyc}} = \sum_{k=1}^{\infty} \frac{x^k}{k} = \log(1/(1-x))$. A permutation can also be decomposed into a set of cycles, so we have $\hat{S} = \hat{E} \circ \widehat{\text{Cyc}}$ which corresponds to the rather trivial identity $1/(1-x) = \exp(\log(1/(1-x)))$. But if we limit what cycles are allowed we can get something a bit less trivial. If we truncate $\widehat{\text{Cyc}}$ after the second term we only allow cycles of length 1 or 2. That way we see that $\exp(x + x^2/2)$ is the egf of involutions. In a similar way we can drop all odd terms or all even terms to get that the egf of set partitions with an even number of odd-size blocks is $\cosh(\sinh(x))$.

Proposition (10)**.** Let F be some structure and F^c be the structure of its connected components, forests and trees for example, permutations and cycles, etc. We can say F is the structure of multisets of F^c. Then $\hat{F}(x) = \exp(\widehat{F^c}(x))$ and $F(x) = \exp(\sum_{k=1}^{\infty} F^c(x^k)/k)$. The k-th summand contributes connected components of size k. If F^c is the ogf of (a_i) this is equivalent to $F(x) = \prod_{k=1}^{\infty}(1-x^k)^{-a_k}$. Inverting these we get $\widehat{F^c}(x) = \log(\hat{F}(x))$ and $F^c(x) = \sum_{k=1}^{\infty} \mu(k)\log(F(x^k))/k$ where μ is the möbius function.

Proposition (11)**.** Let F be the structure of cycles of F^c structures. Then $\hat{F} = \log((1-\widehat{F^c}(x))^{-1})$ and $F(x) = \sum_{k=1}^{\infty} \varphi(k)\log((1-F^c(x^k))^{-1})/k$ where φ is the euler-phi function. Like above the k-th summand contributes F^c structures of size k.

Proposition (12)**.** Let F be the structure of sets of distinct F^c structures. Then $F(x) = \exp(\sum_{k=1}^{\infty}(-1)^{k-1}F^c(x^k)/k)$. Like above the k-th summand contributes F^c structures of size k. If F^c is the ogf of (a_i) this is equivalent to $F(x) = \prod_{k=1}^{\infty}(1+x^k)^{a_k}$.

These relations between F and F^c structures can be considered in greater generality using the tool of cycle index series, covered in Bergeron et. al., but we'll let these suffice here. Out of the three the first one comes up the most often.

As an example we can consider partitions of integers into odd parts and partitions of integers into distinct parts. By letting $F^c(x) = E(x) - 1$ we get that the ogf of partitions into distinct parts is $\prod_{k=1}^{\infty}(1+x^k)$ by (12). Next we can consider partitions into odd parts that the corresponding ogf is $\prod_{k=0}^{\infty}(1-x^{2k+1})^{-1}$ by (10). Substituting $(1+x^k) = (1-x^{2k})/(1-x^k)$ into the first ogf and cancelling gives us that the two ogfs are equal, showing that the number of integer partitions into odd parts and into distinct parts are equal. The cycle construction comes up less often, but we could for example consider cyclic compositions, compositions of integers defined up to cyclic shifts of the summands. Plugging into (11) and calculating out we can get that there are $-1 + n^{-1}\sum_{k|n}\varphi(k)2^{n/k}$ such compositions of n, where the sum s taken over all divisors of n.

For egfs we can define a few more useful operations that aren't applicable to ogfs due to the labelling structure. When interpreting the product $\hat{A}\cdot\hat{B}$ we imagine constructing an A-structure on some subset and a B-structure on the rest, but what if we wish to constrain which structure ends where? This can be done many ways, but one is the following operation.

Proposition (13)**.** For a labelled A-structure with egf $\hat{A}$ and labelled B-structure with egf $\hat{B}$ we define $\hat{A}^{\square}\cdot\hat{B}$ as the egf of the structure of $\hat{A}\cdot\hat{B}$ except we constrain the atom with minimal label to be in the A-structure. This gives us

$$(\hat{A}^{\square}\cdot\hat{B})(x) = \int_0^x (\partial_t\hat{B}(t))C(t)dt$$

Next we will cover three problem solving tricks involving generating functions. The first is referred to in the source as the xd/dx log trick, but I prefer to call it the pointed log trick. This is because in the world of generating functions $x\partial_x A(x)$ is usually called the pointing of $A(x)$. This is because if $A(x)$ is the ogf of (a_i) then $x\partial_x A(x)$ is the ogf of (ia_i) so it's as if one atom in the structure has been pointed, made special and distinct from the rest. Let us apply the pointed log trick to the egf of the Bell numbers found above, telling us that $e^{e^x-1} = \sum_{k=0}^{\infty} b_n/n!x^n$. Taking the logarithm doesn't really make the right hand side any easier to work with, but the trick is in deriving next because $\log(f)' = f'/f$. Thus this gives us

$$e^{e^x-1} = \sum_{k=0}^{\infty}\frac{b_n}{n!}x^n \overset{\log}{\Longrightarrow} e^x - 1 = \log\left(\sum_{k=0}^{\infty}\frac{b_n}{n!}x^n\right) \overset{x\partial_x}{\Longrightarrow} xe^x = \frac{\sum_{k=0}^{\infty}\frac{nb_n}{n!}x^n}{\sum_{k=0}^{\infty}\frac{b_n}{n!}x^n} \Rightarrow \sum_{k=0}^{\infty}\frac{nb_n}{n!}x^n = (xe^x)\sum_{k=0}^{\infty}\frac{b_n}{n!}x^n$$

Thus by (4) we get $nb_n = \sum_{k=1}^{n}\binom{n}{k}(n-k)b_k$ which can be rewritten as $b_n = \sum_k\binom{n-1}{k}b_k$, giving us a simple recurrence for the Bell numbers with very little effort.

The next trick is called the Snake Oil method which can be tremendously useful to evaluate sums. As described in *generatingfunctionology* the method has five steps. First identify the free variable in the sum, say n, and call the sum you're looking to evaluate $f(n)$. Then let $F(x)$ be the ogf of $(f(i))$. Now multiply over the original sum by x^n and sum on n, giving you $F(x)$ as a double sum. Next swap the order of summation and evaluate the (hopefully) simpler inner sum. Lastly identify the coefficients of $F(x)$, giving you the answer. In the examples that follow we will make use of some 'well known' ogfs, which are given just before the problems section.

As the first example consider $f(n) = \sum_{k\geq 0}\binom{k}{n-k}$. Now we sum and swap, giving $F(x) = \sum_n x^n\sum_{k\geq 0}\binom{k}{n-k} = \sum_{k\geq 0}\sum_n\binom{k}{n-k}x^n$. If the coefficient in the inner sum were $n-k$, this would be much easier, but we can do this simply by splitting x up between the sums. Thus we get $F(x) = \sum_{k\geq 0}x^k\sum_n\binom{k}{n-k}x^{n-k}$. Now we can reindex with $r = n-k$ to get $F(x) = \sum_{k\geq 0}x^k\sum_r\binom{k}{r}x^r = \sum_{k\geq 0}x^k(1+x)^k = \sum_{k\geq 0}(x+x^2)^k = (1-x-x^2)^{-1}$. This is easily verified to be the ogf of the fibonacci numbers, finding that $f(n)$ is simply the n-th fibonacci number.

Next consider $f(n) = \sum_{k\geq 0}\binom{n+k}{2k}2^{n-k}$. We follow the steps:

$$F(x) = \sum_n x^n\sum_k\binom{n+k}{2k}2^{n-k} = \sum_k x^k\sum_n\binom{(n-k)+2k}{2k}(2x)^{n-k} = \sum_k x^k\sum_a\binom{a+2k}{2k}(2x)^a$$

$$= \sum_k x^k \left(\frac{1}{1-2x}\right)^{2k+1} = \sum_k \frac{1}{1-2x}\left(\frac{x}{(1-2x)^2}\right)^k = \frac{1}{1-2x}\frac{1}{1-\frac{x}{(1-x)^2}} = \frac{1/3}{1-x} + \frac{2/3}{1-4x} = \sum_n \left(\frac{1}{3}+\frac{2}{3}4^n\right)x^n$$

A fair bit of calculation, but it's all routine steps. And with that we get that $f(n) = (2 \cdot 4^n + 1)/3$. There are even more powerful methods to deal with these kinds of hypergeometric sums, but they won't be covered here. Those interested can read about WZ pairs in *generatingfunctionology*.

The last trick is often called the root of unity filter. Say we want to compute $\sum_{k\geq 0}\binom{1000}{2k}$. We have the binomial theorem, but only want every other entry. So what we do is take $(1+1)^{1000} = \sum_{k\geq 0}\binom{1000}{k}$ and add it to $(1-1)^{1000} = \sum_{k\geq 0}\binom{1000}{k}(-1)^k$. Here all the odd terms cancel, so we get that twice our desired sum is 2^{1000}, so the answer is 2^{999}. The trick here is that $(1^n + (-1)^n)/2$ is 1 when n is even and 0 when n is odd. This can be done similarly for other moduli by using roots of unity, for example $\omega = \exp(2\pi i/3)$ gives $(1^n + \omega^n + \omega^{2n})/3$ which is 1 when n is a multiple of 3 and 0 otherwise. This can be used to compute $\sum_{k\geq 0}\binom{1000}{3k}$ in a similar fashion.

Lastly there are more kinds of generating functions, each changing what the multiplication of generating functions translates to in terms of the sequence. The last one that will be mentioned here are as follows.

Definition. For a sequence (a_i) we will define its *dirichlet generating function* as $\tilde{A}(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$.

We will shorten this to dsgf going forward. Addition and subtraction behave as expected, but one can also derive the following behaviour for other operations.

Proposition (14)**.** If (a_i) has dsgf $\tilde{A}(s)$ and (b_i) has dsgf $\tilde{B}(s)$ then $(\tilde{A}\cdot\tilde{B})(s)$ is the dsgf of $\left(\sum_{d|i} a_d b_{i/d}\right)$ where the sum runs over all divisors d of i.

Dirichlet generating functions are most useful when dealing with multiplicative functions, which are defined as follows.

Definition. We call a function $f : \mathbb{Z}_+ \to \mathbb{Z}$ is called *multiplicative* if for coprime m, n we have $f(mn) = f(m)f(n)$.

This is because one can derive the following.

Proposition (15)**.** Let f be a multiplicative function. Then we have the formal identity

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_p \left(1 + f(p)p^{-s} + f(p^2)p^{-2s} + f(p^3)p^{-3s} + \dots\right)$$

in which the product on the right extends over all prime numbers p.

As may be familiar to some of you, if we take $f = 1$ constant, the function we get on the left hand side is the Riemann-zeta function, denoted $\zeta(s)$. One can show using the identity above that this gives us that the dsgf of $(\mu(i))$ is $\zeta(s)^{-1}$ where μ is the möbius function. This means that $\tilde{A}(s) = \tilde{B}(s)\zeta(s)$ giving $\tilde{A}(s)/\zeta(s) = \tilde{B}(s)$ can be translated into coefficients as

$$a_n = \sum_{d|n} b_d \Rightarrow b_n = \sum_{d|n} \mu\left(\frac{n}{d}\right) a_d$$

which is known as the möbius inversion formula.

Common generating functions:

$$\sum_{k\geq 0} x^k = \frac{1}{1-x}, \sum_{k\geq 1}\frac{x^k}{k} = \log\left(\frac{1}{1-x}\right), \sum_{k\geq 0}\frac{x^k}{k!} = e^x, \sum_{k\geq 0}(-1)^k\frac{x^{2k+1}}{(2k+1)!} = \sin(x), \sum_{k\geq 0}(-1)^k\frac{x^2k}{(2k)!} = \cos(x)$$

$$\sum_k \binom{\alpha}{k} x^k = (1+x)^\alpha, \sum_n \binom{n+k}{n} x^n = \frac{1}{(1-x)^{k+1}}, \sum_{k\geq 0}\frac{B_n x^n}{n!} = \frac{x}{e^x-1}, \sum_{k\geq 0}(-1)^k\frac{x^{2k+1}}{2k+1} = \tan^{-1}(x)$$

$$\sum_k \binom{2k}{k}\frac{x^k}{k+1} = \frac{1-\sqrt{1-4x}}{2x}, \sum_k \binom{2k}{k} x^k = \frac{1}{\sqrt{1-4x}}$$

Here B_n are the Bernoulli numbers.

Problems:

Problems rated on a scale of difficulty from 1 to 5. Problems with several subproblems are rated by their hardest subproblem. These difficulty levels are only my subjective experience with solving the problems, but I hope they give a rough idea at least. Starred problems require results from outside this lecture that the reader might not be familiar with or use material covered only briefly in the lecture. The source of the problems is given after the difficulty. I tried to give the earliest source I could find in each case. A lot of these problems can be solved without generating functions but they can all be solved with generating functions. I encourage the reader to try to solve them using generating functions.

- (1, generatingfunctionology) Find the ogfs and egfs of (i), $(\alpha i+\beta)$, (i^2), $(\alpha i^2+\beta i+\gamma)$, (3^i) and $(5\cdot 7^i-3\cdot 4^i)$.
- (1, generatingfunctionology) Let $A(x)$ and $\hat{A}(x)$ be the ogf and egf of (a_i). Find the ogf and egf of the following sequences in terms of $A(x)$ and $\hat{A}(x)$ respectively: (a_i+c), (αa_i+c), (ia_i) and $a_0, 0, a_2, 0, a_4, 0, \ldots$.
- (1*, generatingfunctionology) Let $a_0=0$ and $a_1=1$ be the required initial values when applicable. Find the ogfs and egfs for the sequences defined by the recurrences $a_{n+1}=3a_n+2$, $a_{n+1}=\alpha a_n+\beta$, $a_{n+2}=2a_{n+1}-a_n$ and $a_{n+1}=a_n/3+1$.
- (2, Sigurður Jens) Compute $\sum_{k=0}^{n}\binom{n}{k}f_k$ where f_k is the k-th fibonacci number.
- (2, Nowhere in particular) Compute $\sum_{k=0}^{n}\binom{n}{k}(k+1)^{-1}$.
- (2, generatingfunctionology) Use dsgfs to prove $\sum_{d|n}\varphi(d)=n$, $\sum_{d|n}\mu(d)=\delta_{n1}$ and $\sum_{d|n}\mu(d)\tau(n/d)=1$. Here δ_{ab} is the Kronecker delta function which is 1 if $a=b$ and 0 otherwise and $\tau(n)$ is the number of divisors of n.
- (2, Yufei Zhao) Show that the number of partitions of n that have no two equal even terms is the same as the number of partitions of n with no four terms equal.
- (2, generatingfunctionology) Find the dsgf of (i), (i^α), $(\log(i))$, $(|\mu(i)|)$ and $(\sum_{d|i}d^q)$.
- (2, Putnam 1957) Let $\alpha(n)$ be the number of compositions of n into parts of 1 and 2. A composition of n is a way to write n as a sum of positive integers where order matters, so $1+2$ and $2+1$ are considered different. Let $\beta(n)$ be the number of compositions of n with parts greater than 1. Show that $\alpha(n)=\beta(n+2)$.
- (2, generatingfunctionology) Let $\lambda:\mathbb{Z}_+\to\mathbb{N}$ such that $\lambda(mn)=\lambda(m)\lambda(n)$ for all m,n, $\lambda(1)=1$ and $\lambda(p)=-1$ for any prime p. Find its dsgf and prove that $\sum_{d|n}\lambda(d)$ is 1 if n is square and 0 otherwise.
- (2, Enumerative Combinatorics) Let $t(n)$ be the number of non-congruent triangles whose sides have integer length and whose perimeter is n, for instance $t(9)=3$, corresponding to $3+3+3$, $2+3+4$ and $1+4+4$. Determine the ogf of $t(n)$.
- (2, generatingfunctionology) Find the egf of permutations with no cycles of length ≤ 3.
- (2, generatingfunctionology) Show that the number of binary words of length n that can't be expressed as the concatenation of more than one identical smaller binary words is given by $\sum_{d|n}\mu(n/d)2^d$.
- (2, Analytic Combinatorics) Find the egf of permutations that result in the identity when composed with themselves f times.
- (3, generatingfunctionology) Evaluate $\sum_k\binom{n+k}{m+2k}\binom{2k}{k}(-1)^k(k+1)^{-1}$.
- (3, Analytic Combinatorics) Prove $np(n)=\sum_{j=1}^{n}\sigma(j)p(n-j)$ where $p(n)$ is the number of partitions of n and $\sigma(n)$ is the sum of the positive divisors of n.
- (3, generatingfunctionology) Evaluate $\sum_k\binom{2n+1}{2p+2k+1}\binom{p+k}{k}$.
- (3, IMO Shortlist 1989) Define (a_i) by $\sum_{d|n}a_d=2^n$. Show that n divides a_n.
- (3, Enumerative Combinatorics) Let $a(n)$ be the number of permutations w that have a square root, i.e. there exists another permutation u such that $u\circ u=w$. Find the corresponding egf. Use it to show $a(2n+1)=(2n+1)a(2n)$.

- (3, generatingfunctionology) In a country that has 1-cent, 2-cent, and 3-cent coins only, show that the number of ways of changing n cents is exactly the integer nearest to $(n+3)^2/12$.
- (3, Evan Chen) Prove that for $n \geq 1$ we have $\sum_{d|n} \tau(d)^3 = (\sum_{d|n} \tau(d))^2$ where $\tau(n)$ is the number of divisors of n.
- (3, Yufei Zhao) Find the number of subsets of $\{1, 2, \ldots, 2007\}$, the sum of whose elements is divisible by 17.
- (3, AMSP 2011 NT3 Exam) Show that $\sum_{k \geq 1} \varphi(k) \lfloor n/k \rfloor = n(n+1)/2$.
- (3, Milan Novakovic) Prove that $\sum_{k=0}^{n} \binom{2n}{2k}\binom{2k}{k} 2^{2n-2k} = \binom{4n}{2n}$.
- (3, Bulgaria 1989) Let $\Omega(n)$ be the number of prime factors of n, counted with multiplicity. Evaluate $\sum_{n=1}^{1989} (-1)^{\Omega(n)} \lfloor 1989/n \rfloor$.
- (3, generatingfunctionology) Let a mountain of coins be an arrangement coins in rows such that the coins in each row form a single block, and that in all rows (except the bottom row) each coin touches exactly two coins from the row beneath it. How many mountains of coins have exactly k coins in the bottom row?
- (4, Euler) Let $P(x)$ be the ogf of the partition numbers. Prove that $P(x)^{-1} = \sum_{k=-\infty}^{\infty} (-1)^k x^{k(3k-1)/2}$.
- (4, Enumerative Combinatorics) Let $f(n)$ denote the number of subsets in $\mathbb{Z}/n\mathbb{Z}$ (the integers modulo n) whose elements sum to 0 in $\mathbb{Z}/n\mathbb{Z}$. Show that $f(n) = n^{-1} \sum_{n|d, d \text{ odd}} \varphi(d) 2^{n/d}$.
- (4, IMO Shortlist 1991) Prove that $\sum_{k=0}^{995} (-1)^k \binom{1991-k}{k} (1991-k)^{-1} = 1991^{-1}$.
- (4*, Analytic Combinatorics) A permutation $\sigma = \sigma_1 \sigma_2 \ldots \sigma_n$ is called alternating if $\sigma_1 > \sigma_2 < \sigma_3 > \sigma_4 < \ldots$. Find the egf for alternating permutations. Hint: Split the problem into considering those of odd length, then those of even length.
- (4, Enumerative Combinatorics) Find the ogf of permutations on $2n-1$ elements such that for $i \neq 2n-1$ if π_i is even then $\pi_i > \pi_{i+1}$ and if π_i is odd then $\pi_i < \pi_{i+1}$.
- (4, IMO 1995) Let p be an odd prime number. How many p-element subsets A of $\{1, 2, \ldots, 2p\}$ are there such that the sums of its elements are divisible by p?
- (4, Leo Moser, Joe Lambek, 1959) Prove that there is a unique way to partition the set of natural numbers in two sets A and B such that for very non-negative integer n (including 0) the number of ways in which n can be written as $a_1 + a_2$, $a_1, a_2 \in A$, $a_1 \neq a_2$ is at least 1 and is equal to the number of ways in which it can be represented as $b_1 + b_2$, $b_1, b_2 \in B$, $b_1 \neq b_2$.
- (5*, Erdös) Given several (at least two, but finitely many) arithmetic progressions, if each natural number belongs to exactly one of them, prove there are two progressions whose common differences are equal.

References:

- Wilf's *generatingfunctionology.*
- Stanley's *Enumerative Combinatorics*, both volumes.
- Sedgewick and Flajolet's *Analytic Combinatorics.*
- *Combinatorial species and tree-like structures* by Bergeron et. al.
- Evan Chen's *Summation* handout, hosted at `https://web.evanchen.cc/handouts/Summation/Summation.pdf`.
- Yufei Zhao's comb3 handout, hosted at `https://yufeizhao.com/olympiad/comb3.pdf`.
- Teaching material from Anders Claesson's course on combinatorics at The University of Iceland.
- Milan Novakovic's *Generating Functions.*